Three-Loop Radiative-Recoil Corrections to Hyperfine Splitting Generated by One-Loop Fermion Factors

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Abstract

We consider three-loop radiative-recoil corrections to hyperfine splitting in muonium generated by diagrams with one-loop radiative photon insertions both in the electron and muon lines. An analytic result for these nonlogarithmic corrections of order $\alpha(Z^2\alpha)(Z\alpha)(m/M)\widetilde{E}_F$ is obtained. This result constitutes a next step in the implementation of the program of reduction of the theoretical uncertainty of hyperfine splitting below 10 Hz.

I. INTRODUCTION

Three-loop radiative-recoil corrections to hyperfine splitting in muonium are enhanced by the large logarithm cubed of the electron-muon mass ratio [1] (see, also review [2]). The leading logarithm cubed contribution is generated by the graphs with insertions of the electron one-loop polarization operators in the two-photon exchange graphs. It may be obtained almost without any calculations by substituting the effective charge $\alpha(M)$ in the leading recoil correction of order $(Z\alpha)(m/M)\widetilde{E}_F$, and expanding the resulting expression in a power series in α. Calculation of the logarithm squared term of order $\alpha^2(Z\alpha)(m/M)\widetilde{E}_F$

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is more challenging [3]. Different graphs generate logarithm squared terms, and all such contributions were obtained a long time ago [1,3,4]. The sum of the logarithm cubed and logarithm squared terms is given by the expression [1]

$$\Delta E = \left(-\frac{4}{3}\ln^3\frac{M}{m} + \frac{4}{3}\ln^2\frac{M}{m}\right)\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\widetilde{E}_F. \tag{2}$$

Due to recent experimental and theoretical progress, single-logarithmic and nonlogarithmic contributions of orders $\alpha^2(Z\alpha)(m/M)\widetilde{E}_F$ and $\alpha(Z^2\alpha)(Z\alpha)(m/M)\widetilde{E}_F$ to hyperfine splitting in muonium are now also phenomenologically relevant. Numerous sets of gauge invariant diagrams generate single-logarithmic and nonlogarithmic contributions.

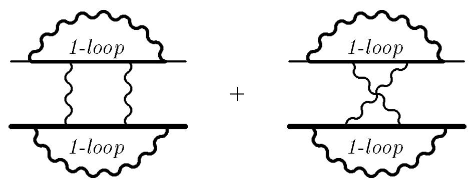

FIG. 1. Diagrams with two fermion factors

Below we consider three-loop radiative-recoil corrections to hyperfine splitting in muonium generated by the diagrams in Fig. 1. These diagrams are obtained from the skeleton diagrams in Fig. 2 by making all one-loop radiative photon insertions both in the electron and muon lines. The two-loop radiative-recoil corrections generated by all one-loop radiative photon insertions only in the electron or only in the muon line were calculated earlier (see, e. g., review in [2]). The leading recoil correction of order $Z\alpha(m/M)\widetilde{E}_F$ is generated by the skeleton diagrams with two exchanged photons in Fig. 2. The characteristic loop momenta in the skeleton diagrams are larger than the electron mass, and therefore the leading recoil correction to hyperfine splitting may be calculated in the scattering approximation, ignoring the wave function momenta of order $mZ\alpha$ (see, e. g., [2]). It was obtained a long time ago [5–7]

$$\Delta E_{rec} = -\frac{3mM}{M^2-m^2}\frac{Z\alpha}{\pi}\ln\frac{M}{m}\ \widetilde{E}_F. \tag{3}$$

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[1] We define the Fermi energy as

$$\widetilde{E}_F = \frac{16}{3}Z^4\alpha^2\frac{m}{M}\left(\frac{m_r}{m}\right)^3 ch\,R_\infty, \tag{1}$$

where m and M are the electron and muon masses, m_r is the reduced mass of the electron-muon system, α is the fine structure constant, c is the velocity of light, h is the Planck constant, R_∞ is the Rydberg constant, and Z is the nucleus charge in terms of the electron charge ($Z=1$ for muonium). The Fermi energy $\widetilde{E}_F$ does not include the muon anomalous magnetic moment a_μ which does not factorize in the case of recoil corrections, and should be considered on the same grounds as other corrections to hyperfine splitting.

Note that this correction is proportional to the logarithm of the electron-muon mass ratio, and quite remarkably it turns out that the logarithmic contribution is a complete result, there is no nonlogarithmic contribution of this order.

The radiative insertions can only increase the characteristic integration momenta in the diagrams in Fig. 2 and hence the scattering approximation remains valid for calculation of two- and three-loop radiative-recoil corrections. The two-loop radiative-recoil corrections of order $\alpha(Z\alpha)(m/M)\widetilde{E}_F$ generated by the one-loop radiative photon insertions in the electron line are logarithmic in the electron-muon mass ratio. Since the leading recoil correction of order $Z\alpha(m/M)\widetilde{E}_F$ is linear in the logarithm of the large mass ratio, one could expect that the correction of order $\alpha(Z\alpha)(m/M)\widetilde{E}_F$ is proportional to the logarithm squared. This does not happen and the logarithm squared contributions cancel as was first discovered in [8] by direct calculation. The simplest way to understand this cancellation is to recall that in the Landau gauge radiative insertions in the electron line are nonlogarithmic [9], and hence, being gauge invariant, the sum of these insertions is nonlogarithmic in any gauge. The single-logarithmic term of order $\alpha(Z\alpha)(m/M)\widetilde{E}_F$ was obtained in [10], and the nonlogarithmic terms were calculated numerically in [11] and analytically in [12]

$$\Delta E = \left[\frac{15}{4}\ln\frac{M}{m} + 6\zeta(3) + 3\pi^2\ln 2 + \frac{\pi^2}{2} + \frac{17}{8}\right]\frac{\alpha(Z\alpha)}{\pi^2}\frac{m}{M}\ \widetilde{E}_F. \tag{4}$$

One more feature of the calculations in [11,12] deserves to be mentioned. One-loop radiative insertions in the electron line include the terms connected with the one-loop anomalous magnetic moment. These terms have different low energy behavior in comparison with all other terms in the dressed electron line and could in principle compilcate calculation of the radiative-recoil corrections. However, as was discovered in [11,12] the terms connected with the one-loop anomalous magnetic moment do not give any contribution at all to the radiative-recoil corrections of order $\alpha(Z\alpha)(m/M)\widetilde{E}_F$. Finally, let us mention that numerically the nonlogarithmic part of the correction of order $\alpha(Z\alpha)(m/M)\widetilde{E}_F$ is rather large, of order π^2, which is just what one should expect for the constants accompanying the large logarithm.

The two-loop radiative-recoil corrections of order $(Z^2\alpha)(Z\alpha)(m/M)\widetilde{E}_F$ are generated by all one-loop radiative photon insertions only in the muon line in the diagrams in Fig. 2, and were obtained in [11,13]

$$\Delta E = \left[\frac{9}{2}\zeta(3) - 3\pi^2\ln 2 + \frac{39}{8}\right]\frac{(Z^2\alpha)(Z\alpha)}{\pi^2}\frac{m}{M}\ \widetilde{E}_F. \tag{5}$$

Two features of these corrections deserve to be mentioned. First, radiative insertions in the muon line do not generate logarithmic terms at all, as can be understood with the help of the generalized low energy theorem [10,14]. Second, just as in the case of the insertions in the electron line the terms connected with the muon anomalous magnetic moment do not give any contribution to the radiative-recoil corrections of order $(Z^2\alpha)(Z\alpha)(m/M)\widetilde{E}_F$.

In this work we analytically calculate three-loop radiative-recoil corrections to hyperfine splitting in muonium generated by the diagrams in Fig. 1 with all one-loop radiative photon insertions both in the electron and muon lines. We show that these corrections are nonlogarithmic and unlike the case of the radiative-recoil corrections of orders $\alpha(Z\alpha)(m/M)\widetilde{E}_F$ and $(Z^2\alpha)(Z\alpha)(m/M)\widetilde{E}_F$ the one-loop anomalous magnetic moments of both particles give nonvanishing contributions to the correction under investigation.

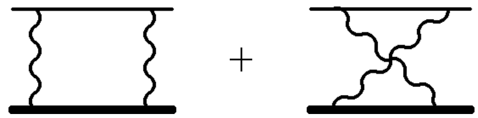

FIG. 2. Diagrams with two-photon exchanges

II. GAUGE INVARIANT REPRESENTATION FOR RADIATIVE CORRECTIONS

Let us consider the general structure of the radiative-recoil corrections in Fig. 1. To this end it is convenient to introduce the one-loop fermion factor $L_{\mu\nu}(k)$ as a sum of the diagrams in Fig. 3. In terms of the electron and muon factors the radiative-recoil contribution to hyperfine splitting generated by the ladder and crossed-ladder diagrams in Fig. 1 has the form

$$\Delta E = -\frac{3}{8}\frac{(Z\alpha)mM}{\pi}\;\widetilde{E}_F\int\frac{d^4k}{i\pi^2(k^2+i0)^2}\Big[L^{(e)}_{\mu\nu}(k)+L^{(e)}_{\nu\mu}(-k)\Big]L^{(\mu)}_{\mu\nu}(-k). \qquad (6)$$

The sum of the electron factors $C^{(e)}_{\mu\nu}(k)\equiv L^{(e)}_{\mu\nu}(k)+L^{(e)}_{\nu\mu}(-k)$ which enters eq.(6) for the radiative corrections is just the gauge invariant Compton scattering amplitude for a virtual photon, and satisfies the identity $k^\mu C^{(e)}_{\mu\nu}(k)=0$. The electron Compton amplitude is invariant under the substitution $k\to -k$ and $\mu\to\nu$, and hence, we can substitute the muon Compton amplitude instead of the muon factor in the integral in eq.(6) $L^{(\mu)}_{\mu\nu}(-k)\to[L^{(\mu)}_{\mu\nu}(-k)+L^{(\mu)}_{\nu\mu}(k)]/2\equiv C^{(\mu)}_{\mu\nu}(-k)/2$ obtaining a more symmetric expression for the energy shift

$$\Delta E = -\frac{3}{16}\frac{(Z\alpha)mM}{\pi}\;\widetilde{E}_F\int\frac{d^4k}{i\pi^2k^4}C^{(e)}_{\mu\nu}(k)C^{(\mu)}_{\mu\nu}(-k). \qquad (7)$$

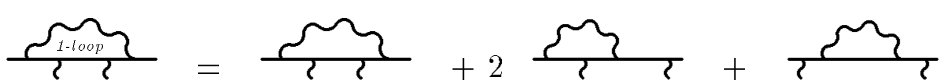

FIG. 3. Fermion factor

To simplify further calculations we represent the electron and muon Compton amplitudes in eq.(7) as sums of two gauge invariant terms (we write the formula only for the electron, and the respective expression for the muon is obtained by the substitution $m\to M$, $\alpha\to Z^2\alpha$)

$$C^{(e)}_{\mu\nu}(k)=C^{(e,a)}_{\mu\nu}(k)+C^{(e,b)}_{\mu\nu}(k), \qquad (8)$$

where

$$C^{(e,a)}_{\mu\nu}(k)=\frac{\alpha}{2\pi}\frac{1}{2m}\Bigg[\frac{\sigma_{\mu\rho}k^\rho(\hat p-\hat k+m)\gamma_\nu\;+\;\gamma_\mu(\hat p-\hat k+m)\sigma_{\nu\rho}(-k^\rho)}{k^2-2mk_0} \qquad (9)$$

$$+\frac{\sigma_{\nu\rho}(-k^\rho)(\hat p+\hat k+m)\gamma_\mu\;+\;\gamma_\nu(\hat p+\hat k+m)\sigma_{\mu\rho}k^\rho}{k^2+2mk_0}\Bigg]$$

corresponds to the anomalous magnetic moment, and $C^{(e,b)}_{\mu\nu}(k)$ includes all other terms.

It is easy to check directly that $C^{(e,a)}_{\mu\nu}(k)$, and hence, $C^{(e,b)}_{\mu\nu}(k)$ are gauge invariant. The breakdown in eq.(8) is helpful because $C^{(e,a)}_{\mu\nu}(k)$, and $C^{(e,b)}_{\mu\nu}(k)$ have different behavior at small photon momenta k. As we will see below this different low energy behavior determines the structure of integrals for the contributions to hyperfine splitting.

We can further simplify the amplitude $C^{(e,a)}_{\mu\nu}(k)$, preserving only the terms which contribute to hyperfine splitting. The simplified expression (still satisfying the Ward identity $k^\mu C^{(e,a)}_{\mu\nu}(k)=0$) has the form

$$C^{(a)}_{\mu\nu}(k)=\frac{\alpha}{\pi}\frac{k^2\,\gamma_\mu\hat{k}\gamma_\nu\ -\ k^2\,k_0\gamma_\mu\gamma_\nu\ +\ k_0\,(k_\mu\hat{k}\gamma_\nu+\gamma_\mu\hat{k}k_\nu)}{k^4-4m^2k_0^2}. \tag{10}$$

In terms of the representation in eq.(8) the contribution to hyperfine splitting in eq.(6) can be written as a sum of three gauge invariant terms

$$\Delta E=-\frac{3}{16}\frac{(Z\alpha)mM}{\pi}\ \widetilde{E}_F\int\frac{d^4k}{i\pi^2k^4}\Big[C^{(e,a)}_{\mu\nu}(k)C^{(\mu,a)}_{\mu\nu}(-k)+C^{(e,b)}_{\mu\nu}(k)C^{(\mu,a)}_{\mu\nu}(-k) \tag{11}$$

$$+C^{(e)}_{\mu\nu}(k)C^{(\mu,b)}_{\mu\nu}(-k)\Big]\equiv\Delta E^I+\Delta E^{II}+\Delta E^{III}.$$

It is important to note that we know in advance that there is no logarithm of the mass ratio in the sum of all contributions in eq.(11). Such a logarithm can only arise from the integration region $m<k<M$, where the electron factor is in the asymptotic regime. The asymptotic expression for the electron factor was calculated e.g., in [3], and contains only the skeleton spinor structure $\gamma_\mu\hat{k}\gamma_\nu$. On the other hand, all terms in the muon factor except the term with the muon anomalous magnetic moment are, in this integration region, additionally suppressed by an extra factor k^2/M^2 in comparison with the logarithmic skeleton integral, and thus cannot produce a logarithmic contribution. As to the term with the muon anomalous magnetic moment, its contribution to the recoil integral vanishes identically due to its spinor structure, see, e. g., [14].

III. TWO ANOMALOUS MAGNETIC MOMENTS

Let us start our calculation with the term ΔE^I connected with the product of two one-loop anomalous magnetic moments. Projecting the spinor structures of the fermion factors, written in the form of eq.(9) or eq.(10), on hyperfine splitting we obtain

$$\Delta E^I=\frac{\alpha(Z^2\alpha)(Z\alpha)mM}{\pi^3}\ \widetilde{E}_F\int\frac{d^4k}{i\pi^2k^4}\frac{k^2(k^4+4k^2k_0^2+k_0^4)}{(k^4-4k_0^2m^2)(k^4-4k_0^2M^2)}, \tag{12}$$

or after the Wick rotation and transition to the spherical coordinates $k_0=k\cos\theta$, $|\mathbf{k}|=k\sin\theta$

$$\Delta E^I=\frac{2\alpha(Z^2\alpha)(Z\alpha)mM}{\pi^4}\ \widetilde{E}_F\int_0^\infty dk^2\int_0^\pi d\theta\sin^2\theta\frac{(1+4\cos^2\theta+\cos^4\theta)}{(k^2+4m^2\cos^2\theta)(k^2+4M^2\cos^2\theta)}. \tag{13}$$

Calculating the angular integral we discover that the remaining momentum integral diverges like dk^2/k^3. This divergence indicates the existence of the nonrecoil correction of order $\alpha(Z^2\alpha)$, which is of lower order in $Z\alpha$. It is connected with the one-photon exchange, and is well known. We subtract this power divergence and, after the subtraction, obtain a convergent integral, which can be easily calculated

$$\Delta E^I = \frac{9}{8}\frac{\alpha(Z^2\alpha)(Z\alpha)}{\pi^3}\frac{mM}{M^2-m^2}\ln\frac{M}{m}\ \widetilde{E}_F. \tag{14}$$

IV. SUBTRACTED ELECTRON FACTOR AND THE MUON ANOMALOUS MAGNETIC MOMENT

The second contribution in eq.(11) arises from the product of the subtracted electron factor and the muon anomalous magnetic moment, and we write it in the form

$$\Delta E^{II} = -\frac{3}{8}\frac{(Z\alpha)mM}{\pi}\ \widetilde{E}_F\int\frac{d^4k}{i\pi^2k^4}L_{\mu\nu}^{(e,b)}(k)C_{\mu\nu}^{(\mu,a)}(-k). \tag{15}$$

The muon Compton amplitude is gauge invariant and satisfies the Ward identity $k^\mu C_{\mu\nu}^{(\mu,a)}(k)=0$. Therefore, we can omit all terms in the subtracted electron factor which are proportional to k_μ. This means that we can use the expression for the subtracted electron factor from [15,16], where all terms proportional to k_μ are thrown away. We represent this electron factor as a sum of seven terms $L_{\mu\nu}^{(e,b)}(k)=\sum_1^7 L_{\mu\nu}^{(i)}(k)$, which are

$$L_{\mu\nu}^{(1)}(k)+L_{\mu\nu}^{(2)}(k)=\frac{\alpha}{4\pi}\langle\gamma_\mu\hat{k}\gamma_\nu\rangle_{(e)}\int_0^1 dx\int_0^x\frac{dy}{(-k^2+2mbk_0+m^2a^2)^3}\left(c_1m^2\mathbf{k}^2+c_2k^4\right), \tag{16}$$

$$L_{\mu\nu}^{(3)}(k)+L_{\mu\nu}^{(4)}(k)=\frac{\alpha}{4\pi}\langle\gamma_\mu\hat{k}\gamma_\nu\rangle_{(e)}\int_0^1 dx\int_0^x\frac{dy}{(-k^2+2mbk_0+m^2a^2)^2}\left(c_3k^2+2c_4mk_0\right),$$

$$L_{\mu\nu}^{(5)}(k)+L_{\mu\nu}^{(6)}(k)=\frac{\alpha}{4\pi}\langle\gamma_\mu\gamma_\nu\rangle_{(e)}\int_0^1 dx\int_0^x\frac{dy}{(-k^2+2mbk_0+m^2a^2)^2}\left(c_5mk^2+2c_6k^2k_0\right),$$

$$L_{\mu\nu}^{(7)}(k)=\frac{\alpha}{4\pi}\langle\gamma_\mu\gamma_\nu\rangle_{(e)}\int_0^1 dx\int_0^x\frac{dy}{-k^2+2mbk_0+m^2a^2}\left(c_7\frac{k^2}{m}\right),$$

where $k^2=k_0^2-\mathbf{k}^2$. Each term in the electron factor corresponds to the respective coefficient function c_i , and explicit expressions for these coefficient functions are collected in Table I. We preserve in the electron factor only the spinor structures $\langle\gamma_\mu\hat{k}\gamma_\nu\rangle_{(e)}$ and $\langle\gamma_\mu\gamma_\nu\rangle_{(e)}$ relevant for hyperfine splitting, and the projection on hyperfine splitting is understood. Auxiliary functions of the Feynman parameters $a(x,y)$ and $b(x,y)$ are defined by the relationships

$$a^2=\frac{x^2}{y(1-y)},\qquad b=\frac{1-x}{1-y}. \tag{17}$$

TABLE I. Coefficients in the Fermion Factor

c_1	$\frac{16}{y(1-y)^3}\Big[\ (1-x)(x-3y)\ \ -\ \ 2y\ln x\ \Big]$
c_2	$\frac{4}{y(1-y)^3}\Big[\ -(1-x)(x-y-2y^2/x)\ \ +\ \ 2(x-4y+4y^2/x)\ln x\ \Big]$
c_3	$\frac{1}{y(1-y)^2}\Big[\ 1-6x-2x^2-(y/x)(26-6y/x-37x-2x^2+12xy+16\ln x)\ \Big]$
c_4	$\frac{1}{y(1-y)^2}\Big(\ 2x-4x^2-5y+7xy\ \Big)$
c_5	$\frac{1}{y(1-y)^2}\Big(\ 6x-3x^2-8y+2xy\ \Big)$
c_6	$-\frac{(1-x)^2(x-y)}{x^2(1-y)^2}$
c_7	$2\frac{1-x}{x}$

The explicit expression for the muon factor can be obtained from the expression for the electron factor by the substitutions $m \to M$ and $\alpha \to Z^2\alpha$.

Taking projection on the hyperfine splitting and contracting the Lorentz indices, we obtain the integral for the contribution to the hyperfine splitting as

$$\Delta E^{II} = \alpha(Z\alpha)(Z^2\alpha)\ \widetilde{E}_F\ \frac{1}{8\pi^3\mu}\int_0^1 dx \int_0^x dy \int \frac{d^4k}{i\pi^2}\frac{1}{(k^2+i0)^2(k^4-\mu^{-2}k_0^2)} \tag{18}$$

$$\Big\{(6k_0^2-2\mathbf{k}^2)\Big[\frac{c_1\mathbf{k}^2+c_2(k^2)^2}{(-k^2+2bk_0+a^2)^3}+\frac{c_3k^2+c_4 2k_0}{(-k^2+2bk_0+a^2)^2}\Big]$$

$$-(6+2\frac{\mathbf{k}^2}{k^2})k_0\Big[\frac{c_5k^2+c_6k^2 2k_0}{(-k^2+2bk_0+a^2)^2}+\frac{c_7k^2}{-k^2+2bk_0+a^2}\Big]\Big\},$$

where $\mu = m/(2M)$ and we rescaled the integration momentum, so that now it is measured in units of the electron mass.

The analytic calculation of the integrals in eq.(18) is one of the more tedious steps in the present paper. These integrals are of the same type as the integrals in [15,16], and we use for calculations the same methods as in those papers. First we represent each integral as a sum of μ-dependent and μ-independent integrals. The μ-independent integrals admit direct analytic calculation. To calculate the μ-dependent integrals we separate the contributions of large and small integration momenta with the help of an auxiliary parameter σ such that $1 \ll \sigma \ll 1/\mu$. In the region of small momenta we use the condition $\mu k \ll 1$ to simplify the integrand, and in the region of large momenta the same goal is achieved with the help of the condition $k \gg 1$. Finally, for $k \sim \sigma$ both conditions on the integration momenta hold simultaneously, so in the sum of low-momenta and high-momenta integrals all σ-dependent terms cancel, and we obtain a σ-independent result for the integral (for more detailed exposition of this method see, e.g., [17]). Here we skip the calculations and present only the final result

$$\Delta E^{II} = \Big[-\frac{9}{8}\ln\frac{M}{m}+\frac{15}{4}\zeta(3)+\frac{27\pi^2}{16}+\frac{3}{2}\Big]\frac{\alpha(Z^2\alpha)(Z\alpha)}{\pi^3}\frac{m}{M}\ \widetilde{E}_F. \tag{19}$$

V. TOTAL ELECTRON FACTOR AND THE SUBTRACTED MUON FACTOR

Consider now the last contribution

$$\Delta E^{III} = -\frac{3}{16}\frac{(Z\alpha)mM}{\pi}\ \widetilde{E}_F\int\frac{d^4k}{i\pi^2k^4}C^{(e)}_{\mu\nu}(k)C^{(\mu,b)}_{\mu\nu}(-k). \tag{20}$$

Due to the generalized low energy theorem the subtracted virtual muon Compton amplitude $C^{(\mu,b)}_{\mu\nu}(-k)$ is suppressed like k^2/M^2 for momenta $k < M$ (see, e.g., [14]). Hence, the recoil correction of first order in the small mass ratio arises from the integration region in eq.(20) where characteristic momenta are of order M. At these high integration momenta only the leading term in the ultraviolet asymptotic expansion of the one-loop electron factor survives

in the integral. This leading term was calculated in [3], and up to the terms proportional to k_μ and/or k_ν has the form

$$C^{(e)}_{\mu\nu}(k) \to \frac{5\alpha}{2\pi}\frac{\gamma_\mu\hat{k}\gamma_\nu}{k^2}. \tag{21}$$

Due to gauge invariance of the subtracted muon factor $k_\mu C^{(\mu,b)}_{\mu\nu}(-k) = k_\nu C^{(\mu,b)}_{\mu\nu}(-k) = 0$, and then the terms in asymptotic expansion of the electron factor which are linear in k_μ and/or k_ν do not give a contribution to the energy shift in eq.(20). Only the term in eq.(21) is relevant for the calculation of the leading recoil correction. In addition further simplifications can be made. The subtracted muon Compton amplitude also can be written as a sum of terms linear in k_μ and/or k_ν and the remaining terms. But it is easy to see that $k_\mu\gamma_\mu\hat{k}\gamma_\nu = k^2\gamma_\mu$ has zero projection on hyperfine splitting, and hence we can omit all terms proportional to k_μ and/or k_ν in the expression for the subtracted muon Compton amplitude in eq.(6). Then the radiative-recoil contribution to hyperfine splitting of order $\alpha(Z^2\alpha)(Z\alpha)E_F$ in eq.(6) reduces to

$$\Delta E^{III} = -\frac{3}{16}\frac{(Z\alpha)mM}{\pi}\frac{5\alpha}{2\pi}\ \widetilde{E}_F\int\frac{d^4k}{i\pi^2k^6}\gamma_\mu\hat{k}\gamma_\nu C^{(\mu,b)}_{\mu\nu}(-k). \tag{22}$$

This last integral is proportional to the integral for radiative-recoil corrections of order $(Z^2\alpha)(Z\alpha)(m/M)E_F$ generated by radiative insertions in the muon line [13]. Let us recall that the leading term in the asymptotic expansion of the skeleton virtual Compton amplitude is

$$C^{(e,skel)}_{\mu\nu}(k) \to -2\frac{\gamma_\mu\hat{k}\gamma_\nu}{k^2}. \tag{23}$$

Comparing this asymptotics with the expression in eq.(21) and using the result of [13] we obtain

$$\Delta E^{III} = \left[-\frac{45}{8}\zeta(3) + \frac{15\pi^2}{4}\ln 2 - \frac{195}{32}\right]\frac{\alpha(Z^2\alpha)(Z\alpha)}{\pi^3}\frac{m}{M}\ \widetilde{E}_F. \tag{24}$$

VI. SUMMARY

The total three-loop radiative-recoil correction to hyperfine splitting in muonium generated by the diagrams in Fig. 1 with one-loop radiative photon insertions both in the electron and muon lines is given by the sum of the contributions in eq.(14), eq.(19), and eq.(24)

$$\Delta E_1 = \left[-\frac{15}{8}\zeta(3) + \frac{15\pi^2}{4}\ln 2 + \frac{27\pi^2}{16} - \frac{147}{32}\right]\frac{\alpha(Z^2\alpha)(Z\alpha)}{\pi^3}\frac{m}{M}\ \ \widetilde{E}_F. \tag{25}$$

Note that, as explained in Section II, single-logarithmic contributions cancelled in this result. We also would like to emphasize that unlike the case of the radiative-recoil corrections generated by the radiative photon insertions only in the electron or only in the muon line, the one-loop anomalous magnetic moments of both particles give nonvanishing contributions in eq.(25).

Some other three-loop radiative-recoil single-logarithmic and nonlogarithmic radiative-recoil corrections were also calculated recently. The corrections of order $\alpha^2(Z\alpha)(m/M)\widetilde{E}_F$ generated by the graphs with two-loop polarization insertions (irreducible and reducible) in the two-photon exchange diagrams were obtained in [18]

$$\Delta E_2 = \Bigl\{-\Bigl[6\zeta(3)+\frac{33}{4}\Bigr]\ln\frac{M}{m}-\frac{97}{8}\zeta(3)-16\mathrm{Li}_4\Bigl(\frac{1}{2}\Bigr)+\frac{2\pi^2}{3}\ln^2 2-\frac{2}{3}\ln^4 2$$

$$+\frac{5\pi^4}{36}-\frac{13\pi^2}{36}-\frac{4495}{432}\Bigr\}\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\ \widetilde{E}_F. \tag{26}$$

Single-logarithmic and nonlogarithmic corrections generated by the diagrams with one-loop polarization insertions in the exchanged photons and radiative photon insertions in the fermion lines were obtained in [19]. These are corrections of orders $\alpha^2(Z\alpha)(m/M)\widetilde{E}_F$ and $\alpha(Z^2\alpha)(Z\alpha)(m/M)\widetilde{E}_F$, and they have the form

$$\Delta E_3 = \Bigl[\frac{22}{3}\ln\frac{M}{m}+7.36110\ (3)\Bigr]\frac{\alpha^2(Z\alpha)}{\pi^3}\frac{m}{M}\ \widetilde{E}_F \tag{27}$$

$$+\Bigl[\Bigl(6\zeta(3)-4\pi^2\ln 2+\frac{13}{2}\Bigr)\ln\frac{M}{m}+22.51939(5)\Bigr]\frac{\alpha(Z^2\alpha)(Z\alpha)}{\pi^3}\frac{m}{M}\ \widetilde{E}_F,$$

Combining all three-loop single-logarithmic and nonlogarithmic corrections to hyperfine splitting in eq.(25), eq.(26), and eq.(27) we obtain ($Z=1$ below)

$$\Delta E_{tot} = \Bigl[\Bigl(-4\pi^2\ln 2+\frac{67}{12}\Bigr)\ln\frac{M}{m}-14\zeta(3)-16\mathrm{Li}_4\Bigl(\frac{1}{2}\Bigr)+\frac{2\pi^2}{3}\ln^2 2 \tag{28}$$

$$+\frac{15\pi^2}{4}\ln 2-\frac{2}{3}\ln^4 2+\frac{5\pi^4}{36}+\frac{191\pi^2}{144}-\frac{12959}{864}+29.88049\ (6)\Bigr]\frac{\alpha^3}{\pi^3}\frac{m}{M}\ \widetilde{E}_F,$$

or

$$\Delta E_{tot} = \Bigl[\Bigl(-4\pi^2\ln 2+\frac{67}{12}\Bigr)\ln\frac{M}{m}+45.0546\Bigr]\frac{\alpha^3}{\pi^3}\frac{m}{M}\ \widetilde{E}_F. \tag{29}$$

Numerically this contribution to the muonium HFS is

$$\Delta E_{tot} = -0.019\ 2\ \mathrm{kHz}. \tag{30}$$

Currently the theoretical accuracy of hyperfine splitting in muonium is about 70 Hz. A realistic goal is to reduce this uncertainty below 10 Hz (see a more detailed discussion in [2,19]). The new contribution obtained in this paper, eq.(25), together with the results of other recent research [16,18–22] makes achievement of this goal closer. Phenomenologically, the improved accuracy of the theory of hyperfine splitting would lead to a reduction of the uncertainty of the value of the electron-muon mass ratio derived from the experimental data [23] on hyperfine splitting (see, e.g., reviews in [2,24]).

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